Chapter 1

Number System

1.0 Introduction to Number Systems

Why so many Number Systems?

Many number systems and counting systems are used. Using the decimal system it is easy to count up to ten fingers, using the just the fingers on two hands.

Another special number system is the binary system used by digital electronic devices. Because digital circuits work on an electrical 'on or off' two state system, a number system based on 2 (i.e. the binary counting system) is much easier for electronic devices to use. However binary is not a natural choice for human counting or calculation.

1.1 Number Systems in Electronics

Number Systems

Definition: Number system is the way to represent a number in different forms.

A fixed number of values can be written with a single numerical character, then a new column is used to count how many times the highest value in the counting system has been reached. The number of numerical values the system uses is called the base of the system. For example, the decimal system has 10 numerical characters and so has a base of 10:

0 1 2 3 4 5 6 7 8 9

For writing numbers greater than 9 a second column is added to the left, and this column has 10 times the value of the column immediately to its right.

Decimal, (base 10)

Decimal has ten values 0 − 9. If larger values than 9 are needed, extra columns are added to the left. Each column value is ten times the value of the column to its right. For example the decimal value twelve is written 12 (1 ten + 2 ones).

Binary, (base 2)

Binary has only two values 0 − 1. If larger values than 1 are needed, extra columns are added to the left. Each column value is now twice the value of the column to its right. For example the decimal value three is written 11 in binary (1 two + 1 one).

Octal, (base 8)

Octal has eight values 0 − 7. If larger values than 7 are needed, extra columns are added to the left. Each column value is now 8 times the value of the column to its right. For example the decimal value eleven is written 13 in octal (1 eight + 3 ones).

Hexadecimal, (base 16)

Hexadecimal has sixteen values 0 − 15, but to keep all these values in a single column, the 16 values (0 to 15) are written as 0 to F, using the letters A to F to represent numbers 10 to 15, so

avoiding the use of a second column. Again, if higher values than 15 (F in hexadecimal) are needed, extra columns to the left are used. Each column value is sixteen times that of the column to its right. For example the decimal value 20 is written 14 in hexadecimal (1 sixteen + 4 ones).

The reason for these differences is because each system has a different base, and the column values in each system increase by multiples of the base number as columns are added to the left.

Table 1.1.1				
Some column values of different number systems				
Decimal	1000	100	10	1
Binary	8	4	2	1
Octal	512	64	8	1
Hexadecimal	4096	256	16	1

For example:

10_{10} represents the decimal value ten. (1 ten + 0 units)

10_{2} represents the binary value two. (1 two + 0 units)

10_{8} represents the octal value eight. (1 eight + 0 units)

10_{16} represents the hexadecimal value sixteen. (1 sixteen + 0 units)

The System Radix

The base of a system, more properly called the RADIX, is the number of different values that can be expressed using a single digit. Therefore the decimal system has a radix of 10, the octal system has a radix of 8, hexadecimal is radix 16, and binary radix 2.

The range of number values in different number systems is shown in Table 1.1.2,

Table 1.1.2			
(Radix 10)	(Radix 2)	(Radix 8)	(Radix 16)
0	0	0	0
1	1	1	1
2		2	2
3		3	3
4		4	4
5		5	5
6		6	6
7		7	7
8			8
9			9
			A
			B
			C
			D
			E
			F

The Radix Point.

When writing a number, the digits used give its value, but the number is ‘scaled’ by its RADIX

POINT.

For example, 456.2_{10} is ten times bigger than 45.62_{10} although the digits are the same.

Notice also that when using multiple number systems, the term 'RADIX point' instead of 'DECIMAL point' is used.

Exponents

A decimal number such as 456.2_{10} can be considered as the sum of the values of its individual dependent on its position within the number (the value of the column):

Table 1.1.3			
Col 2	Col 1	Col 0	Col -1
4 hundreds	+ 5 tens	+ 6 units	+ 2 tenths
(4×10^2)	$+ (5 \times 10^1)$	$+ (6 \times 10^0)$	$+ (2 \times 10_{-1})$
400	+ 50	+ 6	+ 0.2

$$= 456.2_{10}$$

Each digit in the number is multiplied by the system radix raised to a power depending on its position relative to the radix point. This is called the **EXPONENT**. The digit immediately to the left of the radix point has the exponent 0 applied to its radix, and for each place to the left, the exponent increases by one. The first place to the right of the radix point has the exponent -1 and so on, positive exponents to the left of the radix point and negative exponents to the right.

Hexadecimal exponents $98.2_{16} = (9 \times 16^1) + (8 \times 16^0) + (2 \times 16^{-1})$

Octal exponents $56.2_8 = (5 \times 8^1) + (6 \times 8^0) + (2 \times 8^{-1})$

Binary Exponents $10.1_2 = (1 \times 2^1) + (0 \times 2^0) + (1 \times 2^{-1})$

Floating Point Notation

The radix exponent can also be used to eliminate the radix point, without altering the value of the number. In the example below, see how the value remains the same while the radix point moves. It is all done by changing the radix exponent.

$$102.6_{10} = 102.6 \times 10^0 = 10.26 \times 10^1 = 1.026 \times 10^2 = .1026 \times 10^3$$

The radix point is moved one place to the left by increasing the exponent by one.

It is also possible to move the radix point to the right by decreasing the exponent. In this way the radix point can be positioned wherever it is required - in any number system, simply by changing the exponent. This is called FLOATING POINT NOTATION and it is how calculators handle decimal points in calculations.

Normalised Form

By putting the radix point at the front of the number, and keeping it there by changing the exponent, calculations become easier to do electronically, in any radix.

A number written (or stored) in this way, with the radix point at the left of the most significant digit is said to be in NORMALISED FORM. For example $.11011_2 \times 2^3$ is the normalised form of the binary number 110.11_2.

Electronic storage of numbers.

Because numbers in electronic systems are stored as binary digits, and a binary digit can only be 1 or 0, it is not possible to store the radix point within the number. Therefore the number is stored in its normalised form and the exponent is stored separately. The exponent is then reused to restore the radix point to its correct position when the number is displayed.

In electronics systems a single binary digit is called a bit (short for **B**inary Dig**IT**), but as using a single digit would seriously limit the maths that could be performed, binary bits are normally used in groups.

4 bits = 1 nibble

8 bits = 1 byte

Multiple bytes, such as 16 bits, 32 bits, 64 bits are usually called ‘words’, e.g. a 32 bit word. The length of the word depends on how many bits can be physically handled or stored by the system at one time.

4 Bit Binary Representation

Table 1.1.4

Decimal	MSB $2^3 = 8$	4 Bit Binary $2^2 = 4$	$2^1 = 2$	LSB $2^0 = 1$
0	0	0	0	0
1	0	0	0	1
2	0	0	1	0
3	0	0	1	1
4	0	1	0	0
5	0	1	0	1
6	0	1	1	0
7	0	1	1	1
8	1	0	0	0
9	1	0	0	1
10	1	0	1	0
11	1	0	1	1
12	1	1	0	0
13	1	1	0	1
14	1	1	1	0
15	1	1	1	1

When a number is stored in an electronic system, it is stored in a memory location having a fixed number of binary bits. Some of these memory locations are used for general storage whilst others, having some special function, are called **registers**. Wherever a number is stored, it will be held in some form of binary, and must always have a set number of bits. Therefore a decimal number such as 13, which can be expressed in four binary bits as 1101_2 becomes 00001101_2 when stored in an eight-bit register. This is achieved by adding four NON SIGNIFICANT ZEROS to the left of the most significant digit.

Using this system, a binary register that is n bits wide can hold 2^n values.

Therefore an 8 bit register can hold 2^8 values = 256 values (0 to 255)

A 4 bit register can hold 2^4 values = 16 values (0 to 15)

1.2 Converting Between Number Systems

Conversion from any system to decimal.

The number of values that can be expressed by a single digit in any number system is called the system radix, and any value can be expressed in terms of its system radix.

For example the system radix of octal is 8, since any of the 8 values from 0 to 7 can be written as a single digit.

Using the values of each column, (which in an octal integer are powers of 8) the octal value 126_8 can also be written as:

Convert 126_8 to decimal.

$(1x8^2) + (2x8^1) + (6 \text{ x } 8^0)$

$= (1x64)+(2x8)+(6x1) = 86_{10}$

Therefore $126_8 = 86_{10}$.

The same method can be used to convert binary number to decimal:

Convert 1101_2 to decimal.

$= (1x2^3)+(1x2^2)+(0x2^1))+1x2^0)$

$= 8 + 4 +0 +1$

$= 13_{10}$ **Therefore $1101_2 = 13_{10}$**

The same method can be used to convert any system to decimal.

Try these conversions to decimal WITHOUT YOUR CALCULATOR.

110_2

67_8

AFC_{16}

FC_{16}

Converting from Decimal to any Radix

To convert a decimal integer number (a decimal number in which any fractional part is ignored) to any other radix, all that is needed is to continually divide the number by 2 and with each division, write down the remainder, which will be either 1 or 0.

Decimal to Binary

For example, to convert the decimal number 57_{10} to binary:

Divide 57_{10} by the system radix, which when converting to binary is 2. This gives the answer 28, with a remainder of 1.

Continue dividing the answer by 2 and writing down the remainder until the answer = 0

	Remainder
2)57	
2)28	1
2)14	0
2) 7	0
2) 3	1
2) 1	1
2) 0	1

Example 1.2.1 Decimal to Binary Conversion

Now simply write out the remainders, starting from the bottom, to give 111001_2

Therefore $57_{10} = 111001_2$

Decimal to Octal

The same process works to convert decimal to octal, but this time the system radix is 8:

8)57	Remainder
8) 7	1 ↑
2) 0	7

Example 1.2.2 Decimal to Octal Conversion

Therefore $57_{10} = 71_8$

Decimal to Hexadecimal

It also works to convert decimal to hexadecimal, but now the radix is 16:

Therefore $57_{10} = 39_{16}$

16)57	Remainder
16) 3	9 ↑
16) 0	3

Example 1.2.3 Decimal to Hexadecimal Conversion

Numbers with Fractions

Converting the Decimal Integer to Binary

The radix point splits the number into two parts; the part to the left of the radix point is called the INTEGER. The part to the right of the radix point is the FRACTION. A number such as 34.62510 is therefore split into 3410 (the integer), and .62510 (the fraction).

To convert such a fractional decimal number to any other radix, the method described above is used to covert the integer.

2)34	Remainder
2)17	0 ↑
2) 8	1
2) 4	0
2) 2	0
2) 1	0
0	1

Example 1.2.4 Converting the Integer to Binary

So $34_{10} = 100010_2$

Converting the Decimal Fraction to Binary

To convert the fraction, this must be MULTIPLIED by the radix (in this case 2 to convert to binary). Notice that with each multiplication a CARRY is generated from the third column. The Carry will be either 1 or 0 and these are written down at the left hand side of the result. However when each result is multiplied the carry is ignored (don't multiply the carry). Each result is multiplied in this way until the result (ignoring the carry) is 000. Conversion is now complete.

	Carry	
Fraction		625
x Radix		x2
Result	1	250
x Radix		x2
Result	0	500
x Radix		x2
Result	1 ↓	000

Example 1.2.5 Converting the Fraction to Binary

For the converted value just read the carry column from top to bottom.

So $0.625_{10} = .101_2$

Therefore the complete conversion shows that $34.625_{10} = 100010.101_2$

However, with binary, there is a problem in using this method, .625 converted easily but many fractions will not. For example if you try to convert .626 using this method you would find that the binary fraction produced goes on to many, many places without a result of exactly 000 being reached.

With some decimal fractions, using the above method will produce carries with a repeating pattern of ones and zeros, indicating that the binary fraction will carry on infinitely. Many decimal fractions can therefore only be converted to binary with limited accuracy. The number of places after the radix point must be limited, to produce as accurate an approximation as required.

Converting Binary to Decimal

To convert from binary to decimal write down the binary number giving each column its correct 'weighting' i.e. the value of the columns, starting with a value of one for the right hand (least significant column − or LEAST SIGNIFICANT BIT) column. Giving each column twice the value of the previous column as you move left.

Table 1.2.1								
Bit	2^7	2^6	2^5	2^4	2^3	2^2	2^1	2^0
Value (weighting) of each bit	128	64	32	16	8	4	2	1
8 Bit Binary	0	1	0	0	0	0	1	1

Example:

To convert the binary number 01000011_2 to decimal, write down the binary number and assign a 'weighting' to each bit as in Table 1.2.1

Now simply add up the values of each column containing a 1 bit, ignoring any columns containing 0.

Applying the appropriate weighting to 01000011 gives 256 + 64 + 2 + 1 = 67

Therefore: $01000011_2 = 67_{10}$

Convert:

- ✓ 34_{10} to binary.
- ✓ 77_{10} to binary.
- ✓ 1234_{10} to binary.
- ✓ 61325_{10} to binary.

Check your answers by converting the binary back to decimal.

Convert:

- ✓ 1101_2 to decimal.
- ✓ 101101_2 to decimal.
- ✓ 1111101_2 to decimal.
- ✓ 100110_2 to decimal.

Check your answer by converting the decimal back to binary.

Binary and Hexadecimal

Converting between binary and hexadecimal is a much simpler process; hexadecimal is really just a system for displaying binary in a more readable form.

Binary is normally divided into Bytes (of 8 bits) it is convenient for machines but quite difficult for humans to read accurately. Hexadecimal groups each 8-bit byte into two 4-bit nibbles, and assigns a value of between 0 and 15 to each nibble. This reduces the eight 1 or 0 characters of binary into just two characters.

For example:

11101001_2 is split into 2 nibbles 1110_2 and 1001_2 then each nibble is assigned a hexadecimal value between 0 and F.

Decimal	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Hexadecimal	0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F

The bits in the most significant nibble (1110_2) add up to 8+4+2+0 = 14_{10} = E_{16}

The bits in the least significant nibble (1001_2) add up to 8+0+0+1 = 9_{10} = 9_{16}

Therefore 11101001_2 = $E9_{16}$

1.3 Binary Arithmetic

Fig. 1.3.1 Rules for Binary Addition

Binary Addition Rules

Arithmetic rules for binary numbers are quite straightforward, and similar to those used in decimal arithmetic. The rules for addition of binary numbers are:

0 + 0 = 0
0 + 1 = 1
1 + 0 = 1
1 + 1 = (1)0

Notice that in Fig. 1.3.1, 1+1 = (1)0 requires a 'carry' of 1 to the next column. Remember that binary 10_2 = 2_{10} decimal.

Fig. 1.3.2 Simple Binary Addition

Example:

	Decimal	Binary
	2	10
	1 +	01 +
Answer	3	11

Fig. 1.3.3 Binary Addition with Carry

Binary addition is carried out just like decimal, by adding up the columns, starting at the right and working column by column towards the left.

Just as in decimal addition, it is sometimes necessary to use a 'carry', and the carry is added to the next column. For example, in Fig.1.3.3 when two ones in the right-most column are added, the result is 2_{10} or 10_2. The least significant bit of the answer is therefore 0 and the 1 becomes the carry bit to be added to the 1 in the next column.

	Decimal	Binary
	3	0011
	1 +	0001 +
Carry		0110
	4	0100

Fig. 1.3.4 Rules for Binary Subtraction

Binary subtraction rules

The rules for binary subtraction are quite straightforward except that when 1 is subtracted from 0, a borrow must be created from the next most significant column. This borrow is then worth 2_{10} or 10_2 as a 1 bit in the next column to the left is always worth twice the value of the column on its right.

0 - 0 = 0
0 - 1 = 1*
1 - 0 = 1
1 - 1 = 0

*After 10_2 is borrowed from next column on left.

Binary Subtraction

The rules for subtraction of binary numbers are again similar to decimal. When a large digit is to be subtracted from a smaller one, a 'borrow' is taken from the next column to the left. In decimal subtractions the digit 'borrowed in' is worth ten, but in binary subtractions the 'borrowed in' digit must be worth 2_{10} or binary 10_2.

After borrowing from the next column to the left, a 'pay back' must occur.

Fig. 1.3.5 shows how binary subtraction works by subtracting

Decimal	Binary
	Payback Borrow
11	1 ¹011
5 −	0¹ 101 −
6	0 110

5_{10} from 11_{10} in both decimal and binary. Notice that in the third column from the right (2^2) a borrow from the (2^3) column is made and then paid back in the MSB (2^3) column.

Note: In Fig 1.3.5 a borrow is shown as $^{1}0$, and a payback is shown as 0^{1}. Borrowing 1 from the next highest value column to the left converts the 0 in the 2^2 column into 10_2 and paying back 1 from the 2^2 column to the 2^3 adds 1 to that column, converting the 0 to 01_2.

Fig. 1.3.5 Binary Subtraction

Limitations of Binary Arithmetic

Now back to ADDITION to illustrate a problem with binary arithmetic. In Fig. 1.3.6 notice how the carry goes right up to the most significant bit.

This is not a problem with this example as the answer 1010_2 (10_{10}) still fits within 4 bits, but what would happen if the total was greater than 15_{10}?

```
       Binary
        0111
        0011 +
Carry   1110
        1010
```

Fig. 1.3.6 Limits of 4 Bit Arithmetic

As shown in Fig 1.3.7 there are cases where a carry bit is created that will not fit into the 4-bit binary word. When arithmetic is carried out by electronic circuits, storage locations called registers are used , if that can hold only four bits, then this example would raise a problem. The final carry bit is lost because it cannot be accommodated in the 4-bit register, therefore the answer will be wrong.

```
           Binary
            1111
            0001 +
Carry (1)   1110
      (1)   0000
```

Fig. 1.3.7 The Overflow Problem

1.4 Signed Binary

Signed Binary Notation

All the binary arithmetic problems looked yet used only POSITIVE numbers. The reason for this is that it is not possible in PURE binary to signify whether a number is positive or negative.
There are a number of ways in which binary numbers can represent both positive and negative values, One of the simplest of these systems is SIGNED BINARY, also often called 'Sign and Magnitude', is commonly an 8 bit system that uses the MSB to indicate a positive or a negative value. By convention, a 0 in this position indicates that the number given by the remaining 7 bits is positive, and a 1 indicates that the number is negative.

For example:

$+45_{10}$ in signed binary is $(0)0101101_2$

-45_{10} in signed binary is $(1)0101101_2$

Note:

The brackets around the msb (the sign bit) are included here for clarity but brackets are not normally used. Because only 7 bits are used for the actual number, the range of values the system can represent is from -127_{10} or 11111111_2, to $+127_{10}$.

A comparison between signed binary, pure binary and decimal numbers is shown in Table 1.4.1. Notice that in the signed binary representation of positive numbers between $+0_{10}$ and $+127_{10}$, all the positive values are just the same as in pure binary. However the pure binary values equivalents of $+128_{10}$ to $+255_{10}$ are now considered to represent negative values -0 to -127.

This also means that 0_{10} can be represented by 00000000_2 (which is also 0 in pure binary and in decimal) and by 10000000_2 (which is equivalent to 128 in pure binary and in decimal).

Table 1.4.1			
Binary	**Decimal**	**Signed Binary**	
11111111	255	−127	
11111110	254	−126	
11111101	253	−125	
11111100	252	−124	
↑	↑	↑	**−**
10000011	131	−3	
10000010	130	−2	
10000001	129	−1	
10000000	128	−0	
01111111	127	+127	
01111111	126	+126	
01111101	125	+125	
01111100	124	+124	
↑	↑	↑	**+**
00000011	3	+3	
00000010	2	+2	
00000001	1	+1	
00000000	0	+0	

Signed Binary Arithmetic

Because the signed binary system now contains both positive and negative values, Subtraction now becomes possible without the problems of borrow and payback described in Number Systems1.3. However there are still problems. Look at the two examples illustrated in Fig. 1.4.1 and 1.4.2, using signed binary notation.

In Fig. 1.4.1 two positive (msb = 0) numbers are added and the correct answer is obtained. This is really no different to adding two numbers in pure binary as described Number Systems 1.3.

	Decimal	Binary
	7	00000111
	5 +	00000101 +
Carry		00001110
	12	00001100

Fig. 1.4.1 Adding Positive Numbers in Signed Binary

In Fig. 1.4.2 however, the negative number −5 is ad ded to +7, the same action in fact as SUBTRACTING 5 from 7, which means that subtraction should be possible by merely adding a negative number to a positive number. Although this principle works in the decimal version the result using signed binary is 10001100_2 or -12_{10} which of course is wrong, the result of 7 − 5 should be +2.

	Decimal	Binary
	7	00000111
	-5 +	10000101 +
Carry		00001110
	2	10001100

Fig. 1.4.2 Adding Positive & Negative Numbers in Signed Binary

Although signed binary can represent positive and negative numbers, if it is used for calculations, some special action would need to be taken, depending on the sign of the numbers used, and how the two values for 0 are handled, to obtain the correct result. While signed binary does solve the problem of REPRESENTING positive and negative numbers in binary, and to some extent carrying out binary arithmetic, there are better sign and magnitude systems for performing binary arithmetic. These systems are the ONES COMPLEMENT and TWOS COMPLEMENT systems.

1.5 One's and Two's Complement

One's Complement

The complement (or opposite) of +5 is −5. When representing positive and negative numbers in 8-bit one's complement binary form, the positive numbers are the same as in signed binary notation i.e. the numbers 0 to +127 are represented as 00000000_2 to 01111111_2. However, the complement of these numbers, that is their negative counterparts from −128 to −1, are represented by 'complementing' each 1 bit of the positive binary number to 0 and each 0 to 1.

For example:

$+5_{10}$ is 00000101_2 and

-5_{10} is 11111010_2

Notice in the above example, that the most significant bit (msb) in the negative number -5_{10} is 1, just as in signed binary. The remaining 7 bits of the negative number however are not the same as in signed binary notation. They are just the complement of the remaining 7 bits, and these give the value or magnitude of the number.

The problem with the signed binary arithmetic described before was that it gave the wrong answer when adding positive and negative numbers. Does ones complement notation give better results with negative numbers than signed binary?

Fig. 1.5.1 shows the result of adding −4 to +6, using One's complement, this is the same as **subtracting** +4 from +6, so it is crucial to arithmetic.

The result, 00000001_2 is 1_{10} instead of 2_{10}.

This is better than subtraction in signed binary, but it is still not correct. The result should be $+2_{10}$ but the result is +1 (notice that there has also been a carry into the none existent 9th bit).

	Decimal	Binary
	+6	00000110
	-4 +	11111011 +
Carry		(1)11111100
	+2	00000001

Fig. 1.5.1 Adding Positive & Negative Numbers in Ones Complement

Fig. 1.5.2 shows another example, this time adding two negative numbers −4 and −3.Because both numbers are negative, they are first converted to ones complement notation.

$+4_{10}$ is 00000100_2 in pure 8 bit binary, so complementing gives 11111011.

This is -4_{10} in ones complement notation.

	Decimal	Binary
	-4	11111011
	-3 +	11111100 +
Carry		(1)11110000
	- 7	11110111

Fig. 1.5.2 Adding Two Negative Numbers in Ones Complement

+3 is 00000011_{10} in pure 8 bit binary, so complementing gives 11111100.

This is -3_{10} in ones complement notation.

The result of 11110111_2 is in its complemented form so the 7 bits after the sign bit (1110111), should be re-complemented and read as 0001000, which gives the value 8_{10}. As the most significant bit (msb) of the result is 1 the result must be negative, which is correct, but the remaining seven bits give the value of −8. This is still wrong by 1, it should be −7.

End Around Carry

There is a way to correct this however. Whenever the ones complement system handles negative numbers, the result is 1 less than it should be, e.g. 1 instead of 2 and −8 instead of −7, but another thing that happens in negative number ones complement calculations is that a carry is 'left over' after the most significant bits are added. Instead of just disregarding this carry bit, it can be added to the least significant bit of the result to correct the value. This process is called 'end around carry' and corrects for the result −1 effect of the ones complement system.

There are however, still problems with both ones complement and signed binary notation. The one's complement system still has two ways of writing 0_{10}(00000000_2 = +0 and 11111111_2 = -0_2). In any number system, the positive and negative versions of the same number should add to produce zero. As can be seen from Table 1.5.1, adding +45 and −45 in decimal produces a result of zero, but this is not the case in either signed binary or ones complement.

Table 1.5.1			
	Decimal	Signed Binary	Ones Complement
	+45	00101101	00101101
	−45	10101101	11010010
Binary Sum		11011010	11111111
Decimal Sum	0_{10}	-90_{10}	-127_{10}

This is not good enough, however there is a system that overcomes this difficulty and allows correct operation using both positive and negative numbers. This is the Two's Complement system.

Two's Complement Notation

Twos complement notation solves the problem of the relationship between positive and negative numbers, and achieves accurate results in subtractions.

To perform binary subtraction the two's complement system uses the technique of complementing the number to be subtracted. In the ones complement system this produced a result that was 1 less than the correct answer, but this could be corrected by using the 'end around carry' system. This still left the problem that positive and negative versions of the same number did not produce zero when added together.

The process of producing a negative number in Two's Complement Notation is illustrated in Table 1.5.2.

Table 1.5.2 Producing a Twos Complement Negative Number	
+5 in 8-bit binary (or 8-bit Signed Binary) is	00000101
Complementing to produce the Ones Complement	11111010
With 1 added	1
So -5 in Twos Complement is	11111011

This version of −5 now, not only gives the correct answer when used in subtractions but is also the additive inverse of +5 i.e. when added to +5 produces the correct result of 0, as shown in Fig. 1.5.3

	Decimal	Twos Complement Binary
	+5	00000101
	-5 +	11111011 +
Carry		(1)11111110
	0	00000000

Note that in two's complement the (1) carry from the most significant bit is discarded as there is no need for the 'end

around carry' fix.

Fig. 1.5.3 Adding a Number to its Twos Complement Produces Zero

Two's Complement Examples

> **Note:** When working with twos complement it is important to write numbers in their full 8 bit form, since complementing will change any leading 0 bits into 1 bits, which will be included in any calculation. Also during addition, carry bits can extend into leading 0 bits or sign bits, and this can affect the answer in unexpected ways.

Twos Complement Addition

Fig 1.5.4 shows an example of addition using 8 bit twos complement notation. When adding two positive numbers, their sign bits (msb) will both be 0, so the numbers are written and added as a pure 8-bit binary addition.

	Decimal	Twos Complement (Pure)Binary
	12	00001100
	7 +	00000111 +
Carry		00011000
	19	00010011

Fig. 1.5.4 Adding Positive Numbers in Two's Complement

Twos Complement Subtraction

Fig.1.5.5 shows the simplest case of twos complement subtraction where one positive number (the subtrahend) is subtracted from a larger positive number (the minuend).
In this case the minuend is 17_{10} and the subtrahend is 10_{10}.

Decimal	Twos Complement	
17	00010001	Minuend
10 -	11110101	Subtrahend
	1 +	Plus 1
	(1)11100010	Carry
7	00000111	Answer
	Discarded	

Because the minuend is a positive number its sign bit (msb) is 0 and so it can be written as a pure 8 bit binary number.

Fig. 1.5.5 Subtracting a Positive Number from a Larger Positive Number

The subtrahend is to be subtracted from the minuend and so needs to be two's complement and turn +10 into −10.
When these three lines of digits, and any carry 1 bits are added, remembering that in two's complement, any carry from the most significant bit is discarded. The answer (the difference between 17 and 10) is $00000111_2 = 7_{10}$, which is correct. Therefore the twos complement method has provided correct result.

Subtraction with a negative result

Some subtractions will of course produce an answer with a negative value. In Fig. 1.5.6 the result of subtracting 17 from 10 should be -7_{10} but the twos complement answer of 11111001_2 certainly doesn't look like -7_{10}. However the sign bit is indicating correctly that the answer is negative, so in this case the 7 bits indicating the value of the negative answer need to be 'two's complemented'. Now, the answer of 10000111_2 appears which confirms that the original answer was in fact −7 in 8 bit two's complement form.

Decimal	Twos Complement	
10	00001010	Minuend
17 -	11110110	Subtrahend
	1 +	Plus 1
	00011100	Carry
-7	11111001	Negative answer so:
Sign bit = negative	10000110	Complement the 7 value
	1 +	bits & add 1
	10000111	to confirm.

Answer was correct, 11111001 is -7 in 8 bit twos complement.

Fig. 1.5.6 Subtraction Producing a Negative Result

There are some cases where even two's complement will give a wrong answer. In fact there are four

conditions where a wrong answer may crop up:

1. When adding large positive numbers.
2. When adding large negative numbers.
3. When subtracting a large negative number from a large positive number.
4. When subtracting a large positive number from a large negative number.

The problem seems to be with the word 'large'. What is large depends on the size of the digital word the microprocessor uses for calculation. As shown in Table 1.5.3, if the microprocessor uses an 8-bit word, the largest positive number that can appear in the problem OR THE RESULT is $+127_{10}$ and the largest negative number will be -128_{10}. The range of positive values appears to be 1 less than the negative range because 0 is a positive number in twos complement and has only one occurrence (00000000_2) in the whole range of 256_{10} values.

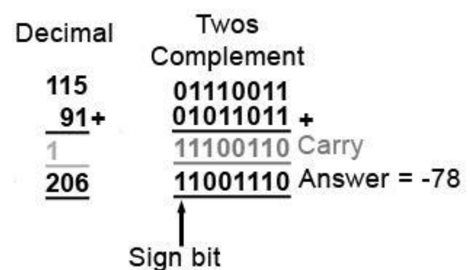
With a 16-bit word length the largest positive and negative numbers will be $+32767_{10}$ and -32768_{10}, but there is still a limit to the largest number that can appear in a single calculation.

Table 1.5.3		
+127	01111111	+
+126	01111110	
+125	01111101	
+2	00000010	
+1	00000001	
0	00000000	
-1	11111111	-
-2	11111110	
-126	10000010	
-127	10000001	
-128	10000000	

Fig. 1.5.7 Carry Overflows into Sign bit

Overflow Problems

A typical overflow problem that can happen even with single byte numbers is illustrated in Fig. 1.5.7.



In this example, the two numbers to be added (115_{10} and 91_{10}) should give a sum of 206_{10} and converting 11001110_2 to decimal looks like the correct answer (206_{10}), but remember that in the 8 bit twos complement system the most significant bit is the sign of the number, therefore the answer appears to be a negative value and reading just the lower 7 bits gives 1001110_2 or -78_{10}. Although twos complement negative answers are not easy to read, this is clearly wrong, as the result of adding two positive numbers must give a positive answer.

According to the information in Fig 1.5.6, as the answer is negative, complementing the lower 7 bits of 11001110_2 and adding 1 should reveal the value of the correct answer, but carrying out the complement+1 on these bits and leaving the msb unchanged gives 10110010_2 which is -50_{10}. This is nothing like the correct answer of 206_{10} so what has happened?

The 8 bit twos complement notation has not worked here because adding 115 + 91 gives a total

greater than +127, the largest value that can be held in 8-bit two's complement notation.

What has happened is that an overflow has occured, due to a 1 being carried from bit 6 to bit 7 (the most significant bit, which is of course the sign bit), this changes the sign of the answer. Additionally it changes the value of the answer by 128_{10} because that would be the value of the msb in pure binary. So the original answer of 78_{10} has 'lost' 128_{10} to the sign bit. The addition would have been correct if the sign bit had been part of the value, however the calculation was done in two's complement notation and the sign bit is not part of the value.

1.6 Binary Codes

Representing Decimal Numbers

If a particular number of binary bits could represent the numbers 0 to 9, but this doesn't happen in pure binary, a 3 bit binary number represents the values 0 to 7 and 4 bit represents 0 to 15. What is needed is a system where a group of binary digits can represent the decimal numbers 0-9, or ten times those values, 10-90 etc.

To make this possible, binary codes are used that have ten values, but where each value is represented by the 1s and 0s of a binary code. These special 'half way' codes are called BINARY CODED DECIMAL or BCD. There are several different BCD codes, but they have a basic similarity. Each of the ten decimal digits 0 to 9 is represented by a group of 4 binary bits.

In fact any ten of the 16 available four bit combinations could be used to represent 10 decimal numbers, and this is where different BCD codes vary. There can be advantages in some specialist applications, For example BCD codes are most often used for the display of decimal digits. The most commonly encountered version of BCD binary code is the BCD_{8421} code. In this version the numbers 0 to 9 are represented by their pure binary equivalents, 4 bits per decimal number, in consecutive order.

BCD Codes

BCD_{8421} code is so called because each of the four bits is given a 'weighting' according to its column value in the bi nary system. The LSB has the weight or value 1, the next bit, going left, the value 2. The next bit has the value 4, and the MSB has the value 8, as shown in Table 1.6.1.

So the 8421_{BCD} code for the decimal number 6_{10} is 0110_{8421}. Check this in Table 1.6.1.

Table 1.6.1

	MSB	BCD_{8421}		LSB
0	0	0	0	0
1	0	0	0	1
2	0	0	1	0
3	0	0	1	1
4	0	1	0	0
5	0	1	0	1
6	0	1	1	0
7	0	1	1	1
8	1	0	0	0
9	1	0	0	1

For numbers greater than 9 the system is extended by using a second block of 4 bits to represent tens and a third block to represent hundreds etc.

24_{10} in 8 bit binary would be 00011000 but in BCD_{8421} is 0010 0100.

992_{10} in 16 bit binary would be 0000001111100000_2 but in BCD_{8421} is 1001 1001 0010.
One of the main drawbacks of BCD is that, because sixteen values are available from four bits, but only ten are used, there are several redundant values whichever BCD system is used.

Convert:

321_{10} to BCD_{8421}

233_{10} to BCD_{8421}

4786_{10} to BCD_{8421}

65231_{10} to BCD_{8421}

Check answers by converting BCD_{8421} back to decimal.

10010001 BCD_{8421} to decimal.

10000011 BCD_{8421} to decimal.

001101110110 BCD_{8421} to decimal.

001100101100011 BCD_{8421} to decimal. BCD_{8421}.

Check answers by converting decimal back to

Display Decoder/Drivers

Depending on the type of display some further code conversion may also be needed. One popular type of decimal display is the 7 segment display used in LED and LCD numerical displays, where any decimal digit is made up of 7 segments arranged as a figure 8, with an extra LED or LCD dot that can be used as a decimal point, as shown in Fig 1.6.1. These displays therefore require 7 inputs, one to each of the LEDs a to g (the decimal point is usually driven separately). Therefore the 4 bit output in BCD must be converted to supply the correct 7 bit pattern of outputs to drive the display.

Fig. 1.6.1 Seven Segment Display

The four BCD bits are usually converted (decoded) to provide the correct logic for driving the 7 inputs of the display by integrated circuits such as the HEF4511B BCD to 7 segment decoder/driver from NXP Semiconductors and the 7466 BCD to 7 segment decoder.

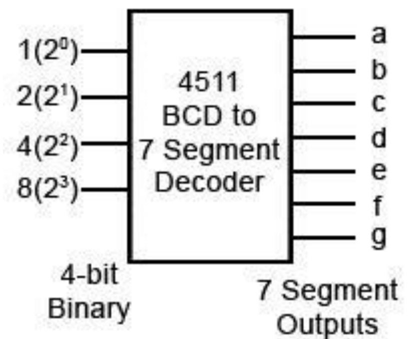


Fig. 1.6.2 Driving a 7 Segment Display

Question

BCD to 7 segment decoders implement a logic truth table such as the one illustrated in Table 1.6.2. There are different types of display implemented by different types of decoder, notice in table 1.6.2 that some of the output digits* may be either 1 or 0 (depending on the IC used). Why would this be, and what effect would it have on the display?

Table 1.6.2											
0	0	0	0	1	1	1	1	1	1	0	0
0	0	0	1	0	1	1	0	0	0	0	1
0	0	1	0	1	1	0	1	1	0	1	2
0	0	1	1	1	1	1	1	0	0	1	3
0	1	0	0	0	1	1	0	0	1	1	4
0	1	0	1	1	0	1	1	0	1	1	5
0	1	1	0	0*	0	1	1	1	1	1	6
0	1	1	1	1	1	1	0	0	0*	0	7
1	0	0	0	1	1	1	1	1	1	1	8
1	0	0	1	1	1	1	0*	0	1	1	9
1	0	1	0	0	0	0	0	0	0	0	Blank
1	0	1	1	0	0	0	0	0	0	0	Blank
1	1	0	0	0	0	0	0	0	0	0	Blank
1	1	0	1	0	0	0	0	0	0	0	Blank
1	1	1	0	0	0	0	0	0	0	0	Blank
1	1	1	1	0	0	0	0	0	0	0	Blank

Notice that the 4 bit input to the decoder illustrated in Table 1.6.2 can, in this case, be in either BCD_{8421} or in 4 bit binary as any binary number over 9 will result in a blank display.

Alternative BCD Codes

Although BCD_{8421} is the most commonly used version of BCD, a number of other codes exist using other values of weighting. Some of the more common variations are shown below. The weighting values in these codes are not randomly chosen, but each has particular merits for specific applications. Some codes are more useful for displaying decimal results with fractions, as with financial data. With others it is easier to assign positive and negative values to numbers. For example with Excess 3 code, 3_{10} is added to the original BCD value and this makes the code 'reflexive', that is the top half of the code is a mirror image and the complement of the bottom half. Other codes are designed to improve error detection in specific systems. Some of these less common BCD codes are shown in Table 1.6.3.

Table 1.6.3

Decimal	7421	5421	5211	2421	Excess 3
0	0000	0000	0000	0000	0011
1	0001	0001	0001	0001	0001
2	0010	0010	0010	0010	0101
3	0011	0011	0101	0011	0110
4	0100	0100	0111	0100	0111
5	0101	1000	1000	0101	1000
6	0110	0110	1001	0110	1001
7	1000	0111	1011	0111	1010
8	1001	1011	1101	1110	1100
9	1010	1100	1111	1111	1100

Gray Code

Binary codes are not only used for data output. Another special binary code that is extensively used for reading positional information on mechanical devices such as rotating shafts is Gray Code. This is a 4 bit code that uses all 16 values, and as the values change through 0-15_{10} the code's binary values change only 1 bit at a time, (see Table 1.6.4). The binary values are encoded onto a rotating disk (Fig. 1.6.3) and as it rotates the light and dark areas are read by optical sensors.

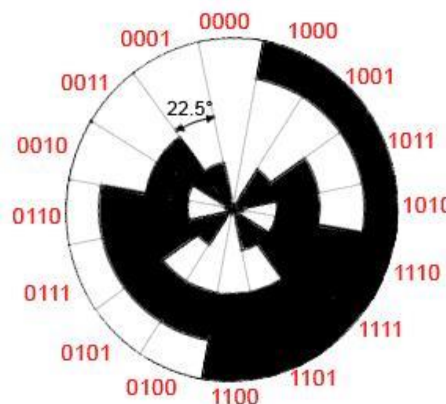


Fig. 1.6.3 Four Bit Gray Code Disk

As only one sensor sees a change at any one time, this reduces errors that may be created as the sensors pass from light to dark (0 to 1) or back again. The problem with this kind of sensing is that if two or more sensors are allowed to change simultaneously, it cannot be guaranteed that the data from the sensors

Table 1.6.4

0	0000
1	0001
2	0011
3	0010
4	0110
5	0111
6	0101
7	0100
8	1100
9;	1101
10	1111
11	1110
12	1010
13	1011
14	1001
15	1000

would change at exactly the same time. If this happened there would be a brief
time when a wrong binary code may be generated, suggesting that the disk is in a different position to its actual position. The one bit at a time feature of Gray Code effectively eliminates such errors. Notice also that the sequence of binary values also rotates continually, with the code for 15 changing back to 0 with only 1 bit changing. With a 4 bit coded disk as illustrated in Fig. 1.6.3, the position is read every 22.5° but with more bits, greater accuracy can be achieved.